Resonance widening in spherical GW detectors: model descriptions of the dissipation processes

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Abstract. Internal friction effects are responsible for line widening of the resonance frequencies in mechanical oscillators and result in damped oscillations of its eigenmodes with a decay time Q/ω. We study the solutions to the equations of motion for the case of spherical oscillators, to be used as next generation of acoustic gravitational wave detectors, based on various different assumptions about the material's constituent equations. Quality factor dependence on mode frequency is determined in each case, and a discussion of its applicability to actual gravitational wave detectors is made on the basis of available experimental evidence.

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1. Introduction

Spherical Gravitational Wave (GW) detectors will almost certainly be, in one of its variants (solid, hollow or dual), the next generation of acoustic antennae, due to their multimode-multifrequency capabilities [1, 2, 3, 4, 5, 6, 7], as well as their potentially enhanced sensitivity relative to their currently operating cylindrical counterparts [8, 9, 10, 11, 12]. Conviction that this is going to be the case has encouraged a remarkable research effort within the GW community, and a variety of important topics have been addressed, ranging from theoretical to practical aspects of the problem. Several countries worldwide are currently developing projects to build and operate spherical detectors [13, 14, 15].

Some of the salient properties of a spherical body as a detector of GWs can be accurately established by means of an idealised model, where dissipative effects can be safely overlooked [16, 17, 18]. Such effects are however very important when it comes to determining the ultimate sensitivity of the device, due to their fundamental bearing on the *noise* characteristics of the antenna via the fluctuation–dissipation theorem, more specifically its *spectral density*.

In a solid elastic body there are many oscillation modes in addition to the fundamental one chosen for GW detection, which therefore add noise to the frequency

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band of operation. It has been suggested [19, 20] that normal mode analysis of this contribution to the noise level can be either inconvenient or too expensive in terms of computational cost if heat is e.g. generated by a laser spot on the oscillating solid, such as happens in an interferometric GW detector or in a dual sphere [7]. Experimental measurements clearly show that this is the case [21].

The naïve approach, and actually the one most often taken, to account for dissipative effects is to add an *ad hoc* term proportional to the velocity in the solid's eigenfrequency equation. This results in a damped oscillatory behaviour of the sort

$$e^{-\omega t/Q} \sin \omega t \tag{1}$$

where the *quality factor* Q is assumed constant (i.e., time independent), and accounts for the linewidth of the mode with frequency ω. It is well known [22] that this quality factor is different for different oscillation modes, so the set of Q's is very large, and the procedure appears highly artificial. On the other hand, there has been criticism of the concept of dissipation forces proportional to velocity in terms of actual noise spectrum in an elastic solid [23], and suggestion that a better description may rather derive from the use of *complex* frequencies.

In the specialised literature on the subject (see [24] and references therein), viscoelastic effects are often described by means of so called *constituent equations*. These are extensions of Hooke's law relating the stress and strain tensors in the solid. Much like in a simple one-dimensional spring, internal friction forces can be considered proportional to the instantaneous velocity of the oscillating mass, hence constituent equations usually contain time derivatives of those tensors, and depend on a small number of viscosity parameters, to be added to the elastic Lamé coefficients λ and μ —see below. There is however no unique way in which this idea can be carried over from a one-dimensional system into a constituent equations set for a three-dimensional solid, and so different alternatives result in different models for the purpose.

In this paper we propose to study and discuss the results of applying to a spherical GW detector the equations of various such phenomenological models, in view to determine how quality factors change from mode to mode in each case. This, it is hoped, will generate some insight into the nature of the viscous processes which take place in a specific spherical GW detector, and help to assess on the basis of spectral measurements which particular class of viscoelastic solid a given material belongs to. In turn, better understanding of material's macroscopic properties should also contribute relevant information to the currently important issue of spherical GW detector design. In section 2 we present the general equations, and in the subsequent sections we successively consider the Kelvin–Voigt, Maxwell, Standard Linear and Genaralised Mechanical models. As we shall see, the sphere's vibration eigenmodes always group into the usual families of *toroidal* (purely torsional) and *spheroidal* modes, but different quality factor dependences arise in different models. In section 6 we match our theoretical results to a number of experimental measurements of quality factors performed on various spheres, and discuss which is the most appropriate model to fit such data. In the final section we present a summary of conclusions, and two appendices are added to clarify a few mathematical technicalities.

2. The general equations

GWs bathing the Earth are known to be extremely weak —see e.g. [25] for a review— so that the classical equations of linear Elasticity are very good to describe the GW

induced motions of a spherical antenna in the expected frequency range, roughly 10^2–10^4 Hz. These equations are [26]

$$\rho \frac{\partial^2 s_i}{\partial t^2} - \frac{\partial \sigma_{ij}}{\partial x_j} = f_i \tag{2}$$

where $\mathbf{s}(\mathbf{x}, t)$ is the field of displacements in the elastic body, and $\mathbf{f}(\mathbf{x}, t)$ is the GW induced density of forces acting on the solid —see [16] for full technical details. σ_{ij} is the the *stress* tensor, and is related to the *strain* tensor

$$s_{ij} \equiv \frac{1}{2}\left(\frac{\partial s_i}{\partial x_j} + \frac{\partial s_j}{\partial x_i}\right) \tag{3}$$

through a set of *constituent equations*. In the case of a non-dissipative (ideal) solid these are simply the expression of Hooke's law. In a dissipative one, such equations include *time derivatives* of both s_{ij} and σ_{ij} to account for internal friction effects. Constituent equations are of the following general type:

$$L(s_{ij}, \dot{s}_{ij}, \ddot{s}_{ij}, \ldots; \sigma_{ij}, \dot{\sigma}_{ij}, \ddot{\sigma}_{ij}, \ldots) = 0 \tag{4}$$

In this paper we shall limit ourselves to *linear* constituent equations, an excellent approach for a GW detector, as already stressed. In the simplest instance, only first order derivatives will appear in (4), and we shall consider this first. Then we shall also devote some attention to more complicted models.

The equations of motion (2) must be supplemented with suitable *boundary conditions*. We shall prescribe the usual ones

$$\sigma_{ij}\, n_j = 0 \qquad \text{at} \qquad r = R \tag{5}$$

where $\mathbf{n}$ is a unit outward-pointing vector, expressing that the surface of the sphere is free from any tensions and/or tractions.

3. Kelvin–Voigt model

This model assumes that the solid is homogeneous and isotropic, and is characterised by the following constituent equations [24]:

$$\sigma_{ij} = \left(\lambda + \lambda' \frac{\partial}{\partial t}\right) s_{kk}\, \delta_{ij} + 2\left(\mu + \mu' \frac{\partial}{\partial t}\right) s_{ij} \tag{6}$$

The constants λ and μ are the usual Lamé coefficients describing the purely elastic behaviour of the body [26], while the positive coefficients λ' and μ' parametrise its *viscous* properties‖, which are proportional to the change rate of the strain tensor, $\partial_t s_{ij}$.

If equations (6) are replaced into (2) we obtain the equations of motion for $\mathbf{s}(\mathbf{x}, t)$:

$$\rho \frac{\partial^2 \mathbf{s}}{\partial t^2} = \left(\mu + \mu' \frac{\partial}{\partial t}\right) \nabla^2 \mathbf{s} - \left[(\lambda + \mu) + (\lambda' + \mu') \frac{\partial}{\partial t}\right] \nabla(\nabla \cdot \mathbf{s}) + \mathbf{f}(\mathbf{x}, t) \tag{7}$$

The solution to this system of coupled equations can be expressed in terms of a Green function, the construction of which requires explicit knowledge of its *eigenmode* solutions. As is well known, the latter correspond to the free oscillations of the solid

‖ These coefficients are actually analogous to those which describe the viscosity of fluids in Hydrodynamics: *shear* viscosity (μ') and *bulk* viscosity ($2\mu' + 3\lambda'$).

—no density of external forces in the rhs of (7). As usual, we attempt to find such eigen-solutions in the factorised form

$$\mathbf{s}(\mathbf{x},t) = T(t)\,\mathbf{s}(\mathbf{x}) \tag{8}$$

which results in the following (dots on symbols stand for time derivatives):

$$\begin{aligned}\rho\ddot{T}(t)\,\mathbf{s}(\mathbf{x}) = &\left[\mu T(t) + \mu'\dot{T}(t)\right]\nabla^2\mathbf{s}(\mathbf{x}) \\ &+ \left[(\lambda+\mu)T(t) + (\lambda'+\mu')\dot{T}(t)\right]\nabla\left(\nabla\cdot\mathbf{s}(\mathbf{x})\right)\end{aligned} \tag{9}$$

Next we use the well known decomposition of a three dimensional vector field into its irrotational a divergence free components [26]:

$$\mathbf{s}(\mathbf{x}) = \mathbf{s}_l(\mathbf{x}) + \mathbf{s}_t(\mathbf{x})\ , \qquad \nabla\cdot\mathbf{s}_t(\mathbf{x}) = \nabla\times\mathbf{s}_l(\mathbf{x}) = 0 \tag{10}$$

By the methods described in Appendix A, it can be seen that the equations satisfied by $\mathbf{s}_t(\mathbf{x})$, $\mathbf{s}_l(\mathbf{x})$, and $T(t)$ are (A.5)–(A.8), i.e.,

$$\nabla^2\mathbf{s}_t + \mathcal{K}^2\,\mathbf{s}_t = 0 \tag{11a}$$

$$\mu\,T + \mu'\,\dot{T} + \mathcal{K}^{-2}\ddot{T} = 0 \tag{11b}$$

and

$$\nabla^2\mathbf{s}_l + \mathcal{Q}^2\,\mathbf{s}_l = 0 \tag{12a}$$

$$\mu\,T + \mu'\,\dot{T} + \mathcal{Q}^{-2}\ddot{T} = 0 \tag{12b}$$

Since $T(t)$ must fulfil *both* equations (11b) and (12b), the *separation constants* $\mathcal{K}^2$ and $\mathcal{Q}^2$ are *not* independent. The binding relationship is established after it is realised that the solution to those equations is of the form

$$T(t) = e^{\gamma t} \tag{13}$$

where γ is in general a *complex* quantity. We readily obtain

$$\mathcal{Q}^2 = -\,\rho\gamma^2\left[\lambda + 2\mu + \gamma\,(\lambda' + 2\mu')\right]^{-1} \tag{14a}$$

$$\mathcal{K}^2 = -\,\rho\gamma^2\left[\mu + \gamma\,\mu'\right]^{-1} \tag{14b}$$

The values γ can possibly take on are determined by the *boundary conditions*, equations (5). The reader is referred to Appendix B for a detailed description of the eigenvalue algebra of this problem. Just as in the non-dissipative case, there are seen to be two families of eigenmodes: *toroidal* (purely torsional) and *spheroidal*. Viscous effetcs are however small in practice, as inferred from the narrow *linewidth* of the measured resonances. This means that the following inequalities hold in any cases of interest to us:

$$\frac{\mu'}{\mu},\ \frac{\lambda'}{\mu} \ll \frac{1}{\omega} \tag{15}$$

where ω is the frequency of the mode considered. We shall use these inequalities to estimate the roots of the eigenvalue equation (B.15) (see Appendix B) *perturbatively* from the non-dissipative ones, already known —see [16] for full details. Clearly thus, our procedure will be valid for the lower frequency modes. We proceed sequentially for the two families of eigenmodes.

3.1. Toroidal modes

These correspond to the solutions to (see (B.15))

$$\beta_1(\mathcal{K}R) = 0 \tag{16}$$

which is *formally* identical to the toroidal eigenvalue equation for a non-dissipative solid sphere [16]. If we call k^T_{nl} the toroidal wave numbers of the latter, we have

$$\mathcal{K}^T_{nl} = k^T_{nl} = \sqrt{\frac{\rho}{\mu}}\omega^T_{nl} \tag{17}$$

and hence, by equation (14*b*),

$$\gamma^T_{nl} = -(\omega^T_{nl})^2\,\frac{\mu'}{2\mu} + i\omega^T_{nl}\sqrt{1-\left(\frac{\omega^T_{nl}\mu'}{2\mu}\right)} \simeq i\omega - \omega^2\,\frac{\mu'}{2\mu} \tag{18}$$

where the last approximation depends on the validity of the assumption (15). Expression (18) nicely shows how this Kelvin-Voigt model predicts *exponentially damped* eigenmode oscillations. If we recall that such damping is expediently described in terms of a *quality factor* Q —see (1) above— then we discover that the prediction of the model is that

$$Q^T_{nl} = \frac{2\mu}{\mu'}\left(\omega^T_{nl}\right)^{-1} \tag{19}$$

i.e., the quality factor for toroidal modes is *inversely proportional* to the frequency of the mode. The *amplitudes* of these modes have the form

$$\mathbf{s}^T_{KV}(\mathbf{x},t) = \mathbf{s}^T_E(\mathbf{x},t)\,e^{-\omega t/Q}\ , \qquad Q = \frac{2\mu}{\mu'\omega} \tag{20}$$

where the subindex KV stands for 'Kelvin-Voigt', while E refers to the standard frictionless case, whose amplitudes are those given e.g. in reference [16].

3.2. Spheroidal modes

A second alternative to find a non-trivial solution of the linear system (B.9) is to impose the condition

$$\beta_4\left(\mathcal{Q}R, \frac{\lambda+\gamma\lambda'}{\mu+\gamma\mu'}\right)\,\beta_3(\mathcal{K}R) - l(l+1)\,\beta_1(\mathcal{Q}R)\beta_1(\mathcal{K}R) = 0 \tag{21}$$

This is characteristic of the *spheroidal* eigenmodes. By virtue of equations (14*a*) and (14*b*), this relationship can be translated into a condition to be fufilled by γ, and which depends on the ratios λ/μ, λ'/λ and μ'/μ, as well as on the multipole index l. In this case, as we are not dealing with an eigenvalue problem of a selfadjoint operator, complex solutions to equation (21) are allowed —indeed, expected. An exact solution of that equation implies a separation of its real and imaginary parts, followed by numerical calculations which determine the angular frequency and quality factor of the quasinormal mode at hand. We are interested in materials with long decay times, so we shall set up a perturbative solution to equation (21), using

$$\epsilon \equiv \frac{\mu'\omega}{\mu} \tag{22}$$

as the small perturbative parameter. In other words, we assume that the approximation (15) holds. Here, ω stands for a generic spheroidal eigenfrequency of the non-dissipative solid. Obviously, the unperturbed solution, i.e., that corresponding to $\epsilon = 0$, is the elastic solid's solution, already discussed in reference [16]. We thus introduce the perturbative expansion

$$\gamma = \gamma_0 + \gamma_1 \epsilon + O(\epsilon^2), \qquad \gamma_0 = -i\omega, \tag{23}$$

Using equations (14*a*), (14*b*) and (23), we obtain perturbative expansions for the parameters $\mathcal{K}$ and $\mathcal{Q}$ which can be written as

$$\mathcal{K} = k_0 + k_1 \epsilon + O(\epsilon^2), \qquad \mathcal{Q} = q_0 + q_1 \epsilon + O(\epsilon^2), \tag{24}$$

where

$$k_0 = k = \omega \sqrt{\frac{\rho}{\mu}}\,, \qquad k_1 = i \sqrt{\frac{\rho}{\mu}} \left(\gamma_1 + \frac{\omega}{2} \right) \tag{25a}$$

$$q_0 = q = \omega \sqrt{\frac{\rho}{\lambda + 2\mu}}\,, \qquad q_1 = i \sqrt{\frac{\rho}{\lambda + 2\mu}} \left(\gamma_1 + \frac{h' + 2}{h + 2} \frac{\omega}{2} \right) \tag{25b}$$

In the above equations, k and q are the parameters appearing in the elastic sphere's case, and we have introduced the dimensionless ratios

$$h \equiv \frac{\lambda}{\mu}\,, \qquad h' \equiv \frac{\lambda'}{\mu'} \tag{26}$$

which are both zero order quantities. We can now perform the perturbative expansion of the eigenvalue equation. In order to ease the resulting expressions, let us introduce the following notation:

$$\begin{aligned}
l(l+1)\,\beta_1(\mathcal{K}R) &= l(l+1)\beta_1(kR) + l(l+1)\beta_1'(kR) k_1 R\,\epsilon \\
&\equiv B_0 + B_1 k_1 R\,\epsilon && (27a) \\
\beta_1(\mathcal{Q}R) &= \beta_1(qR) + \beta_1'(qR) q_1 R\,\epsilon \\
&\equiv C_0 + C_1 q_1 R\,\epsilon && (27b) \\
\beta_3(\mathcal{K}R) &= \beta_3(kR) + \beta_3'(kR) k_1 R\,\epsilon \\
&\equiv D_0 + D_1 k_1 R\,\epsilon && (27c) \\
\beta_4\left(\mathcal{Q}R, \frac{\lambda + \gamma\lambda'}{\mu + \gamma\mu'}\right) &= \beta_4(qR, h) + \left[\beta_4'(qR, \lambda/\mu) q_1 R - \frac{i}{2}(h' - h)\, j_l(qR) \right] \epsilon \\
&\equiv A_0 + (A_1 q_1 R - i A_1')\,\epsilon && (27d)
\end{aligned}$$

where a prime on a β function denotes differentiation with respect to its *first* argument. We note that the functions $A_0, \ldots, D_1$ (defined in every second line of the above set of equations) are real. With this notation, the zero-th order form of equation (21) is

$$A_0 D_0 - C_0 B_0 = 0 \tag{28}$$

which is simply the condition that ω be a spheroidal eigenvalue of the purely elastic case. On the other hand, the first order expansion of (21) yields

$$(A_0 D_1 - C_0 B_1) k_1 R + (A_1 D_0 - C_1 B_0) q_1 R = i A_1' D_0, \tag{29}$$

whence, using the form of k_1 and q_1, the value of γ_1 ensues. It can be written as

$$\gamma_1 = -\frac{\omega}{2}\,[f(kR,h,h')]^{-1} \tag{30}$$

where we define the dimensionless function f by¶

$$f(kR,h,h') \equiv \\ -\frac{A_0D_1 - C_0B_1 + (A_1D_0 - C_1B_0)(h+2)^{-1/2}}{2A_1'D_0(kR)^{-1} - A_0D_1 + C_0B_1 - (A_1D_0 - C_1B_0)\,(h'+2)(h+2)^{-3/2}} \tag{34}$$

We note that the first order correction obtained for γ is *real*. Therefore, to this order of approximation, the frequencies of vibration remain unaltered, and are the same as those obtained for the elastic solid. Moreover, k_1 and q_1 happen to be purely imaginary. Therefore, the modulus of the radial functions appearing in the spatial part of spheroidal quasinormal modes of vibration will also be the same as those of the ideally elastic solid, for the corrections to k and q will just introduce, to first order, a *complex phase* factor.

Summing up, we have shown that while the spheroidal normal modes of vibration of an elastic solid are given by an expression of the form [16]

$$\mathbf{s}_E^P(\mathbf{x},t) = e^{i\omega_{nl}^P t}\,[A_n l(r)\,Y_{lm}(\theta,\varphi)\,\mathbf{n} - B_{nl}(r)\,i\mathbf{n}\times\mathbf{L}Y_{lm}(\theta,\varphi)] \tag{35}$$

the spheroidal quasinormal modes $\mathbf{s}_{KV}^P$ of a Kelvin–Voigt solid are obtained from the normal modes of the elastic solid according to the following

$$\mathbf{s}_{KV}^P(\mathbf{x},t) = e^{i\omega_{nl}^P t - \omega_{nl}^P t/Q}\,\Big[e^{i\chi_1(r)}A_{nl}(r)Y_{lm}(\theta,\varphi)\,\mathbf{n} \\ -\;\; e^{i\chi_2(r)}B_{nl}(r)\,i\mathbf{n}\times\mathbf{L}Y_{lm}(\theta,\varphi)\Big] \tag{36}$$

the qualitity factor being given by

$$Q_{nl}^P = \frac{2\mu}{\mu'\omega_{nl}^P}\,f(k_{nl}^P R,h,h') \tag{37}$$

where, it is recalled, f is given by (34) as a function of the mode and the coefficients characterising the viscoelastic properties of the solid. The real phases $\chi_{1,2}(r)$ can be computed from equations (B.6)-(B.8), (25a), and (25b). Nevertheless, the explicit (and cumbersome) form of these phases is largely irrelevant, and whe shall omit its explicit

¶ The case in which f takes its simplest form is that of monopole modes. We know (see [16]) that when $l=0$ equation (21) is no longer valid, and must be replaced by

$$\beta_4\left(\mathcal{Q}R,\frac{\lambda+\gamma\lambda'}{\mu+\gamma\mu'}\right) = 0 \tag{31}$$

Using now the expansion (27d) and the fact that $A_0 = 0$ for the uperturbed monopole eigenfrequencies, we obtain for the first order correction to $\mathcal{Q}$,

$$q_1 = iA_1'A_1^{-1} \tag{32}$$

and therefore, using the notation of equation (30) and the relationships (25a)-(25b),

$$f(qR,h,h') = \left[\frac{h'+2}{h+2} - \frac{2\,A_1'}{qR\,A_1}\right]^{-1}. \tag{33}$$

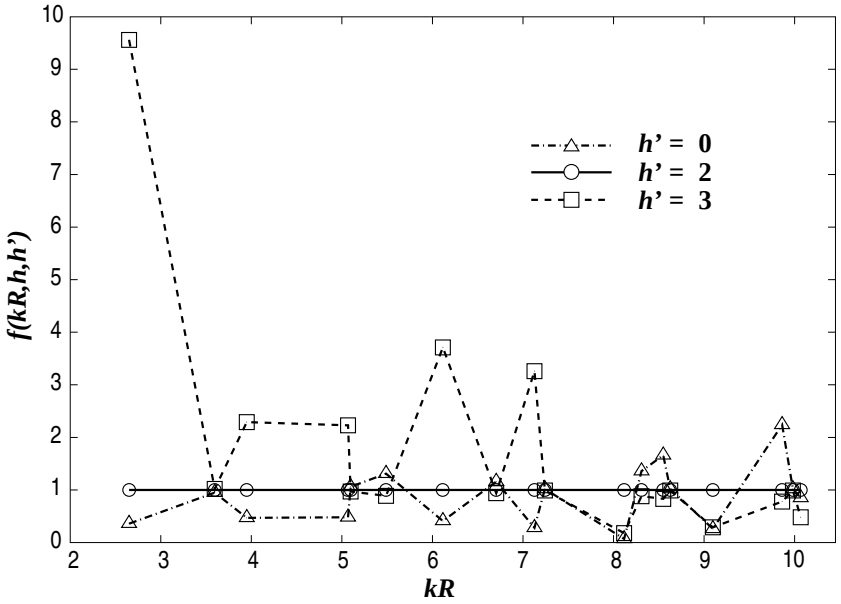

Figure 1. Plot of the function $f(kR, h, h')$ —see (34)— for a Poisson ratio $\sigma = 1/3$ (which means $h = 2$), and for the first 18 (spheroidal) modes of the sphere's spectrum and a few values of the viscoelastic ratio h'. Note that $f(kR, h, h') \equiv 1$ if $h = h'$.

form here [27]. They merely introduce a position dependent shift in the phase of the vibrations which is of order ϵ, therefore not likely to give rise to measurable effects. More interesting and physically relevant is the behaviour of the function f giving the precise dependence of the quality factor on frequency. First of all, it is easily seen that, for the special case $h = h'$, f is equal to 1, and thus the quality factor is proportional to ω^{-1}, as was the case with toroidal modes. But when the aforementioned equality does not hold, numerical calculations are needed. Figure 1 shows that this function has a rather irregular dependence on frequency. We have represented f for the first 18 eigenvalues of the spheroidal spectrum, where the inequality (15) safely holds. It is clear from equation (34) that holding fixed h and kR leaves us with a *linear* function of h', whose slope varies from root to root.

4. Maxwell model

In this section we shall consider constituent equations given by the so called Maxwell model. Like the Kelvin-Voigt, these equations only involve first order time derivatives. As we shall see, quite different predictions will be obtained for the quality factor dependence on the mode frequencies.

The Maxwell model is also isotropic and homogenous, and is characterised by the following constituent equations [24]:

$$\partial_t \sigma_{ij} + \alpha\, \sigma_{kk} \delta_{ij} + \beta\, \sigma_{ij} = \frac{\partial}{\partial t} \left(\lambda\, s_{kk}\, \delta_{ij} + 2\mu\, s_{ij} \right) \tag{38}$$

Here, the constants λ and μ are again the Lamé coefficients describing the elastic behaviour of the body, while the constants α and β parametrise the effects due to

internal friction. To construct factorised solutions, we must also factorise both stress and strain[+]:

$$\sigma_{ij}(\mathbf{x},t) = e^{\gamma t}\,\sigma_{ij}(\mathbf{x})\ , \qquad s_i(\mathbf{x},t) = e^{\gamma t}\,s_i(\mathbf{x}), \tag{39}$$

The constituent equation is thus written, after separation of variables and contraction of its free indices, as

$$(\gamma + 3\alpha + \beta)\,\sigma_{jj}(\mathbf{x}) = \gamma\,(2\mu + 3\lambda)\,s_{jj}(\mathbf{x}) \tag{40}$$

and hence we have the following relationship between the spatial parts of stress and strain:

$$\left(1 + \frac{\beta}{\gamma}\right)\,\sigma_{ij}(\mathbf{x}) = \left(\lambda - \alpha\frac{2\mu + 3\lambda}{\gamma + 3\alpha + \beta}\right)\,s_{kk}(\mathbf{x})\delta_{ij} + 2\mu\,s_{ij}(\mathbf{x}) \tag{41}$$

Like in the Kelvin-Voigt model, we shall be mainly interested in the case of small internal friction, so we shall assume

$$\frac{\beta}{|\,\gamma\,|},\ \frac{\alpha}{|\,\gamma\,|} \ll 1 \tag{42}$$

whence the following constituent relationship results:

$$\sigma_{ij}(\mathbf{x}) = \lambda\left(1 - \frac{\delta}{\gamma}\right)\,s_{kk}(\mathbf{x})\delta_{ij} + 2\mu\left(1 - \frac{\beta}{\gamma}\right)\,s_{ij}(\mathbf{x}) \tag{43}$$

where we have introduced a new constant, δ, defined by

$$\delta \equiv \frac{2\mu + 3\lambda}{\lambda}\,\alpha + \beta, \tag{44}$$

so that we can take as the parameters characterizing the Maxwell solid the set consisting of the Lamé coefficients λ, μ, and the parameters β, δ which describe internal friction.

Let us compare equation (43) with that of the Kelvin–Voigt model —cf. (6)— once the separation of variables has been performed:

$$\sigma_{ij}(\mathbf{x}) = (\lambda + \gamma\lambda')\,s_{kk}(\mathbf{x})\,\delta_{ij} + 2(\mu + \gamma\mu')\,s_{ij}(\mathbf{x}) \tag{45}$$

Comparing equations (43) and (45), we observe that the solution of the Maxwell model can be carried out, as regards the spatial part of $\mathbf{s}$, following the same method used in the previous section for the Kelvin–Voigt model. In fact, we can directly take the expressions there derived, and make the substitutions

$$\mu' \ \longrightarrow\ -\mu\beta\gamma^{-2}\ , \qquad \lambda' \ \longrightarrow\ -\lambda\delta\gamma^{-2} \tag{46}$$

which transform equation (45) into (43). Thus the form of the solutions and boundary conditions for a Maxwell viscoelastic sphere are those of Appendix B with constants $\mathcal{K}$ and $\mathcal{Q}$ now given by the following functions of the parameter γ:

$$\mathcal{Q} = i\sqrt{\frac{\rho}{\lambda + 2\mu}}\,\left[\gamma + \frac{1}{h+2}\left(\frac{h}{2}\delta + \beta\right)\right] \tag{47a}$$

$$\mathcal{K} = i\sqrt{\frac{\rho}{\mu}}\left(\gamma + \frac{\beta}{2}\right) \tag{47b}$$

where the approximation (42) has been taken into account, and $h \equiv \lambda/\mu$.

Thus the two families of quasinormal modes of vibration are also present in this model, and we describe them in the following subsections.

[+] Due to the equations of motion (2), if $\mathbf{s}(\mathbf{x},t)$ is assumed to be separable in the fashion (8) then the strain tensor is separable too, and due to the constituent equation it is easily seen that the only possible time dependence is of the form $\exp(\gamma t)$.

4.1. Toroidal modes

As already discussed, the allowed values for γ are again those making the linear system (B.9) compatible, and there are two alternative ways to accomplish this. The first possibility yields purely tangential ($C_t = C_l = 0$) vibrations satisfying once more the condition

$$\beta_1(\mathcal{K}R) = 0 \quad \Longrightarrow \quad \mathcal{K} = \sqrt{\frac{\rho}{\mu}}\,\omega_{nl}^T \tag{48}$$

ω_{nl}^T being a toroidal eigenfrequency of the elastic sphere. Using the relationship between γ and $\mathcal{K}$ for a Maxwell solid given by equations (47*a*)-(47*b*), we obtain the allowed values γ_{nl}^T as

$$\gamma_{nl}^T = -i\omega_{nl}^T - \frac{\beta}{2} \tag{49}$$

Again, toroidal quasinormal modes have two fundamental properties: they have the same set of eigenfrequencies as the elastic sphere (to first order in the parameters describing internal friction, β in this case), and also exactly the same spatial part (for *all* values of the viscosity parameters). The only difference between Kelvin–Voigt and Maxwell solids as regards toroidal modes appears in the dependence of the quality factor on ω: as equation (49) shows, the quality factor in a linear Maxwell solid increases linearly with frequency. We can express all these properties by means of the following formulæ:

$$\mathbf{s}_M^T(\mathbf{x},t) = \mathbf{s}_E^T(\mathbf{x},t)\, e^{-\omega_{nl}^T t/Q}\ , \qquad Q_{nl}^T = \frac{2\omega_{nl}^T}{\beta} \tag{50}$$

relating Maxwell quasi-normal modes of vibration, $\mathbf{s}_M^T(\mathbf{x},t)$, to elastic normal modes, $\mathbf{s}_E^T(\mathbf{x},t)$, for the toroidal family.

4.2. Spheroidal modes

In order to handle the spheroidal family, we shall resort again to the perturbative expansions already used in the Kelvin–Voigt case, and also in the toroidal family, just described. The Maxwell model trivially reduces to the perfect elastic case when $\beta = \delta = 0$, hence we can take as the perturbative parameter

$$\epsilon = \frac{\beta}{\omega} \tag{51}$$

where ω is the elastic eigenfrequency to which γ approaches when both β and δ approach zero. Perturbative expansions in the fashion of section 3.2 can now be introduced:

$$\gamma = -i\omega + \gamma_1\epsilon\ , \qquad \mathcal{K} = k + k_1\epsilon\ , \qquad \mathcal{Q} = q + q_1\epsilon \tag{52}$$

where the first order corrections k_1 and q_1 are given by equations (47*a*)-(47*b*) as functions of γ_1:

$$k_1 = i\sqrt{\frac{\rho}{\mu}}\left(\gamma_1 + \frac{\omega}{2}\right) \tag{53a}$$

$$q_1 = i\sqrt{\frac{\rho}{\lambda+2\mu}}\left(\gamma_1 + \frac{h'+2}{h+2}\frac{\omega}{2}\right) \tag{53b}$$

The zero–order ratio h' is now given by

$$h' = \frac{\alpha}{\beta}\, h \tag{54}$$

With this definition, together with that of the perturbative parameter, the expressions at hand are formally identical to those of the Kelvin–Voigt model, and therefore the solutions to the Maxwell model share all their properties with their Kelvin–Voigt counterparts; the exception is the dependence of the quality factor on frequency: the product $\gamma_1\epsilon$, which gives the exponential decay, is now *independent of* ω.

Summing up, spheroidal quasinormal modes of the Maxwell solid, $\mathbf{s}^P_M(\mathbf{x},t)$, are related to spheroidal normal modes of a perfectly elastic sphere by the equations

$$\mathbf{s}^P_M(\mathbf{x},t) = \mathbf{s}^P_E(\mathbf{x},t)\, e^{-\omega^P_{nl}t/Q+\chi(r)}\ , \qquad Q^P_{nl} = \frac{2\omega^P_{nl}}{\beta}\, f(k^P_{nl}R,h,h') \tag{55}$$

where the function $f(kR,h,h')$ is again given by (34).

As we see, the only difference between the behaviour of Maxwell and Kelvin–Voigt viscoelastic solids, when the internal friction effects can be considered small, appears in the dependence of Q on frequency. We must however stress that, under other conditions (e.g. static load), both models show larger divergences in their physical properties [28].

5. Other Models

In this section we review other models which have been proposed to address the dynamics of a viscoelastic solid. They are generalisations of those in the two previous sections. We shall however not attempt to find complete solutions to all of them, as it eventually becomes too cumbersome. We shall however discuss in this section some of their most relevant traits.

5.1. The Standard Linear Model

The Standard Linear Model (SLM) for a viscoelastic solid is a generalised combination of the Kelvin–Voigt and Maxwell models. The constituent equations take here the form:

$$\sigma_{ij} + \frac{\partial}{\partial t}\left(\alpha\,\sigma_{kk}\delta_{ij} + 2\beta\,\sigma_{ij}\right) = \left(\lambda + \alpha'\frac{\partial}{\partial t}\right) s_{kk}\delta_{ij} + 2\left(\mu + \beta'\frac{\partial}{\partial t}\right) s_{ij} \tag{56}$$

where the effects of internal friction are described in this case with the aid of four constant parameters: α, β, α', and β'. When looking for factorised solutions, the equations of motion and the above relationship force a time dependence of the form $e^{\gamma t}$ for both stress and strain. When such a dependence is introduced in equation (56), we obtain the following relationship between the spatial parts of the stress and strain tensors:

$$\begin{aligned}(1+2\gamma\beta)\,\sigma_{ij}(\mathbf{x}) = &\left[\lambda + \alpha'\gamma - \alpha\gamma\,\frac{3\lambda+2\mu+\gamma(3\alpha'+2\beta)'}{1+\gamma(3\alpha+2\beta)}\right] s_{kk}(\mathbf{x})\,\delta_{ij} \\ &+ 2(\mu+\beta'\gamma)\,s_{ij}(\mathbf{x})\end{aligned} \tag{57}$$

The case of small internal friction is treated by first order approximation in the quantities parametrising viscous processes, i.e.,

$$\alpha,\ \beta,\ \alpha',\ \beta' \ \ll \mid \gamma \mid^{-1} \tag{58}$$

When such approximation is made, the above equation reduces to

$$\sigma_{ij}(\mathbf{x}) = (\lambda + \lambda'\gamma)s_{kk}(\mathbf{x})\delta_{ij} + 2(\mu + \mu'\gamma)s_{ij}(\mathbf{x}) \tag{59}$$

where we have introduced two new constants given by

$$\mu' \equiv \beta' - 2\beta\mu \ , \qquad \lambda' \equiv \alpha' - 2\lambda\beta - (3\lambda + 2\mu)\alpha \tag{60}$$

Therefore when equation (58) holds, the SLM reduces to a Kelvin–Voigt model, i.e., for small internal friction, both models have the same set of quasinormal modes of vibration, which are characterized by the constants λ, μ, λ' and μ', the latter being given, for the Standard Linear solid, by equations (60).

5.2. Generalised mechanical models

The models analyzed so far are the simplest ones obtained by three dimensional generalisations of mechanical viscoelastic models composed of linear springs. They give rise to differential constituent relations, with time derivatives up to the first order. Considering more involved generalisations yields differential relations involving higher order time derivatives of strain and stress —see e.g. [29]. Thus, quite independently of any reference to the underlying mechanical model, we can consider general differential relations between stress and strain including any number of time derivatives. To ease the formulation of such differential constituent equations for the case of isotropic and homogenous solids, we shall introduce the trace-free parts of the strain and stress tensors, s'_{ij}, σ'_{ij} (usually termed *deviatoric* components in the literature on viscoelasticy [29]), and their traces, s and σ (*dilational* components), defined by

$$s'_{ij} = s_{ij} - \frac{1}{3}\, s\, \delta_{ij} \ , \qquad s \equiv s_{kk} \tag{61a}$$

$$\sigma'_{ij} = s_{ij} - \frac{1}{3}\sigma\delta_{ij} \ , \qquad \sigma \equiv \sigma_{kk}. \tag{61b}$$

In terms of the above quantities, the linear Hooke law for an elastic solid takes the form

$$\sigma = (3\lambda + 2\mu)s \qquad \sigma'_{ij} = 2\mu\, s'_{ij} \tag{62}$$

while the constituent equation of an SLM is written

$$\left[1 + (3\alpha + 2\beta)\,\frac{\partial}{\partial t}\right]\,\sigma = \left[(3\lambda + 2\mu) + (3\alpha' + 2\beta')\,\frac{\partial}{\partial t}\right]\,s \tag{63a}$$

$$\left(1 + 2\beta\,\frac{\partial}{\partial t}\right)\,\sigma'_{ij} = \left(2\mu + 2\beta'\,\frac{\partial}{\partial t}\right)\,s'_{ij} \tag{63b}$$

This equation can now be generalised to include higher order time derivatives. We can thus consider viscoelastic models whose constituent equation is given by

$$R(\partial_t)\sigma = S(\partial_t)s \ , \qquad R'(\partial_t)\sigma'_{ij} = S'(\partial_t)s'_{ij} \tag{64}$$

where we have introduced formal polynomials

$$R(x) \equiv \sum_{l=0}^{N-1} r_l x^l \ , \quad R'(x) \equiv \sum_{l=0}^{N-1} r'_l x^l \tag{65a}$$

$$S(x) \equiv \sum_{l=0}^{N-1} s_l x^l \ , \quad S'(x) \equiv \sum_{l=0}^{N-1} s'_l x^l \tag{65b}$$

so that a general differential model is given for each set of $4N$ real constants r_l, r'_l, s_l, and s'_l characterising the solid. Some of these constants may vanish. The Kelvin-Voigt, Maxwell and SL models considered above are of course special cases within this general class. Several procedures have been proposed in the literature to solve the general equations —see [30] for a review and further reference—, which can be applied to the solid viscoelastic sphere problem. We now sketch how they work in this case of our interest.

Let $\tilde{\sigma}_{ij}(\mathbf{x}, \Omega)$ and $\tilde{s}_{ij}(\mathbf{x}, \Omega)$ be the Fourier transforms of the stress and strain tensors, and $\tilde{s}_i(\mathbf{x}, \Omega)$ that of the displacement vector field:

$$\tilde{\sigma}_{ij}(\mathbf{x}, \Omega) \equiv \int_{-\infty}^{\infty} \sigma_{ij}(\mathbf{x}, t)\, e^{i\Omega t}\, dt \tag{66a}$$

$$\tilde{s}_i(\mathbf{x}, \Omega) \ \equiv \int_{-\infty}^{\infty} s_i(\mathbf{x}, t)\, e^{i\Omega t}\, dt \tag{66b}$$

$$\tilde{s}_{ij}(\mathbf{x}, \Omega) \equiv \int_{-\infty}^{\infty} s_{ij}(\mathbf{x}, t)\, e^{i\Omega t}\, dt \tag{66c}$$

In terms of these, Fourier transforms the constituent equations (64) read:

$$\tilde{\sigma} = \frac{R(i\Omega)}{S(i\Omega)}\, \tilde{s} \ , \qquad \tilde{\sigma}'_{ij} = \frac{R'(i\Omega)}{S'(i\Omega)}\, \tilde{s}_{ij} \tag{67}$$

while the equations of motion are

$$-\,\Omega^2 \tilde{\mathbf{s}} = \frac{1}{3}\left(\frac{R(i\Omega)}{S(i\Omega)} - \frac{R'(i\Omega)}{S'(i\Omega)}\right) \nabla(\nabla \cdot \tilde{\mathbf{s}}) + \frac{R'(i\Omega)}{2S'(i\Omega)}\, \nabla^2 \tilde{\mathbf{s}} \tag{68}$$

Comparing the above equations with the corresponding ones for normal modes of vibration of elastic solids, and the constituent relation (67) with 62, we note that the problem of finding solutions to the equation of motion of a general viscoelastic differential model reduces to that of finding the normal modes of vibration of an elastic solid having *complex* Lamé coefficients given by

$$\tilde{\lambda}(\Omega) = \frac{1}{3}\left(\frac{R(i\Omega)}{S(i\Omega)} - \frac{R'(i\Omega)}{S'(i\Omega)}\right) \ , \qquad \tilde{\mu}(\Omega) = \frac{1}{2}\frac{R'(i\Omega)}{S'(i\Omega)} \tag{69}$$

where the allowed values of Ω are obtained as the solutions to the elastic solid's eigenfrequency equation when the above complex coefficients are used instead of the real, constant Lamé coefficients λ, μ. Generally, Ω will have complex values, thus giving rise to damped system oscillations. After solving for Ω, the spatial part of the solutions is obtained from that of the normal modes by simply substituting the old, real–valued constants ω, λ and μ by the new complex values Ω, $\tilde{\lambda}$ and $\tilde{\mu}$. This method for solving the viscoelasticity is often termed in the literature on the subject

the *Correspondence Principle* [30], and as a matter of fact our previous derivations of the form of the quasinormal modes for Kelvin-Voigt, Maxwell and SL models can be seen to be special cases of its application. The method is applicable to any boundary value problem whose elastic counterpart is solvable. The case of small internal friction (i.e., first order approximation in the coefficients of the polynomials $(65a)$-$(65b)$ has been considered by Graffi [30] for one dimensional wave propagation.

The three dimensional spherical case is also solvable, as we know. The *toroidal* modes are relatively straightforward to obtain from their elastic counterparts due to the simple form of their eigenvalue equation, while the spheroidal ones demand more complex algebra, which becomes increasingly cumbersome as the order N of the model increases. We shall present here the general solution for the toroidal modes for any differential viscoelastic model, whereby we shall obtain the dependence of their Q on frequency. This will also be the approximate dependence for the spheroidal modes, if friction effects are small, as was the case with the first order models analysed so far. A complete solution for the latter modes can also be systematically found, but will be omitted due to its scarcely useful algebraic complexity [27].

5.2.1. Toroidal modes As discussed above, the boundary equation for the toroidal modes in a general viscoelastic model is obtained from the eigenvalue equation of the elastic model:

$$\beta_1(kR) = 0 \ , \qquad k = \sqrt{\frac{\rho}{\mu}}\,\omega \tag{70}$$

Upon substitution of μ by $\tilde{\mu}$, we obtain

$$\beta_1(\mathcal{K}R) = 0 \ , \qquad \mathcal{K} = \sqrt{\frac{2\rho S'(i\Omega)}{R'(i\Omega)}}\,\Omega \tag{71}$$

We know that the only solutions to the eigenvalue equation 70 are the *real* eigenfrequencies of the elastic sphere ω_{nl}^T, and therefore the allowed values for Ω are given by the implicit relationship

$$\sqrt{\frac{2\mu\, S'(i\Omega)}{R'(i\Omega)}}\,\Omega = \omega_{nl}^T \tag{72}$$

Let us now write the polynomials S' and R' in the form

$$S'(x) = 1 + \epsilon \sum_{l=1}^{N-1} s'_l x^l \ , \qquad R'(x) = 2\mu \left(1 + \epsilon \sum_{l=1}^{N-1} r'_l x^l \right) \tag{73}$$

as a suitable one to imply that internal friction effects are small, letting

$$\epsilon \ll 1 \tag{74}$$

The quantities $r'_l\omega^l$ and $s'_l\omega^l$ are thus zero order in ϵ and dimensionless, ω being a toroidal eigenvalue of the elastic case. We then introduce an expansion for Ω in the small parameter ϵ, whose zeroth order term corresponds to a given toroidal eigenfrequency ω of the elastic solid:

$$\Omega = \omega + \epsilon\Omega_1 \tag{75}$$

Under the above conditions, we have

$$\frac{2S'(i\Omega)}{\mu R'(i\Omega)} = 1 + 2i\epsilon \sum_{l=0}^{N-1} t_l \omega^l \ , \qquad t_l = i^{l-1}(s'_l - r'_l)/2 \tag{76}$$

and the value of Ω_1 follows when introducing the above expansion into equation (72), yielding

$$\Omega_1 = \sum_{l=1}^{N-1} t_l \omega^{l+1} \tag{77}$$

In terms of Ω_1, the quality factor reads

$$Q = -\frac{\omega}{\epsilon} \frac{1}{\mathrm{Im}[\Omega_1]} \tag{78}$$

where Im[·] denotes the imaginary part of its argument. Thus we observe that, as regards toroidal modes, using a general differential model gives us a *polynomial* in ω for $1/Q$, with no independent term, so that constant Q is not allowed by these models. The polynomial only contains *odd* powers of the unperturbed frequency ω. In general, whenever $t_l \neq 0$ for even l, the real part of Ω_1 will not vanish, and the angular frequency of the periodic component of the quasinormal modes shall undergo first order corrections. Hence, in order to preserve the elastic spectrum to first order, our model must satisfy the conditions

$$t_l = 0 \qquad (l \text{ even}) \tag{79}$$

Provided the preceeding equation holds, the corrections to $\mathcal{K}$ will be purely imaginary, and therfore the modulus of the spatial part of the modes will remain unaltered, the only effect of viscosity being the addition of a point dependent phase in the fashion of equation (55)*.

The calculation for the spheroidal quasinormal modes can be performed along the same lines but, as we have seen, the algebra is considerably more involved already in the simplest models. It does naturally become more cumbersome as the order of the model increases, so we omit a detailed discussion of its technicalities here.

6. Experimental data

We now present a confrontation of the above theoretical models with available experimental measurements, some of them published [31], and others unpublished as of this date [32].

It is very important, in order to rightly assess the suitability of the theoretical models, to keep in mind that measurements of Q are intrinsically difficult. The most important source of problems is the fact that almost any experimental environment conditions (suspensions, readout, electronics, etc.) significantly couple to the primary oscillator (the sphere itself), and almost invariably result in a *degradation* of the measured Q. The way and intensity in which this happens is however utterly unpredictable. As a rule, theoretical Q values should thus be expected *larger* than measured ones.

* This correction will only appear provided that $N \geq 3$. This is why it was absent in the toroidal families of the previously discussed models.

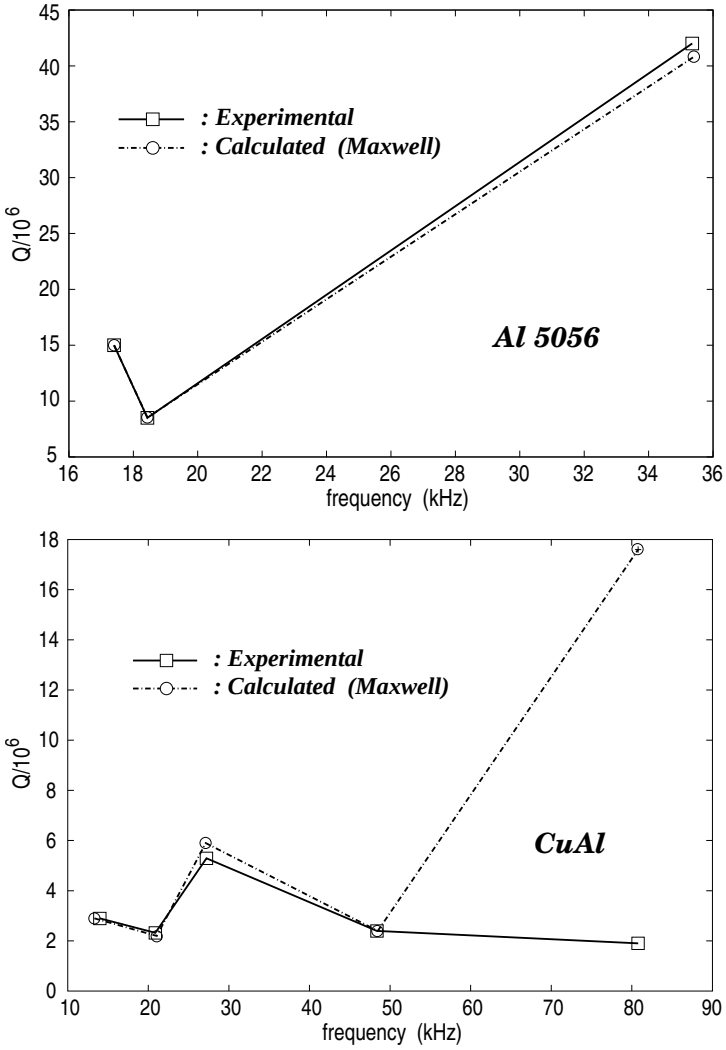

Figure 2. Quality factors of a sample sphere of Al 5056 (top), and a 7 % alloy of CuAl (bottom). The first had a diameter of 153 mm, and measurements were taken at 80 mK, while the second was 150 mm in diameter and had a temperature of 20 mK. As can be appreciated, the Q's of the lower frequency modes match the theoretical predictions fairly well, assuming a Maxwell model with viscoelastic ratios $h' = 1$ and $h' = 0.08$, respectively. The higher mode in CuAl, however, considerably deviates from calculations.

Then, the mode frequencies slightly split up as an unavoidable consequence of the necessity to *suspend* the sample in the laboratory —a fact not considered in the above models. This, at times, generates a forest of frequency peaks when different multipole modes happen to be close to one another. For example, within a mere 0.8 % of their nominal value, we find the three fundamental frequencies: ω^P_{22}, ω^P_{14}, and ω^T_{14} which, if split up, result in as many as 23 Fourier peaks in a very reduced frequency interval. On the other hand, however, the Q's of the various multiplet components

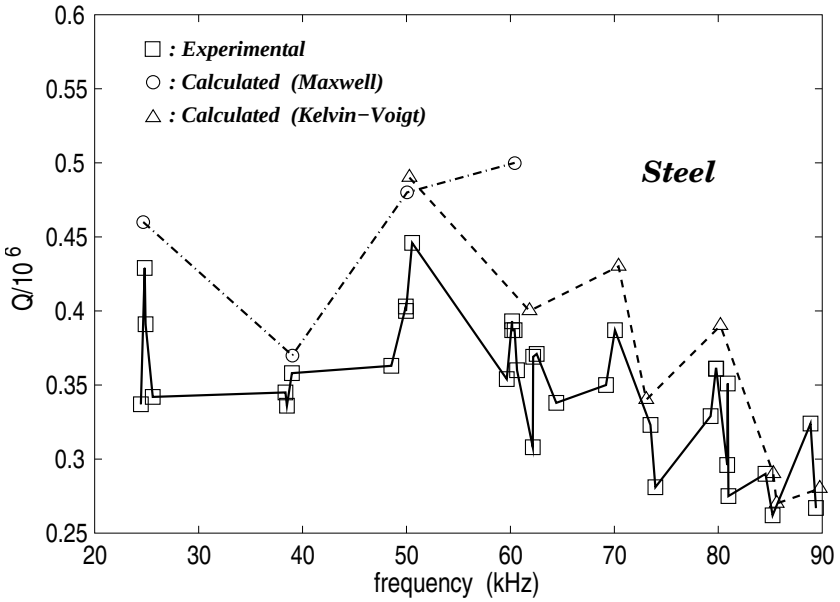

Figure 3. Quality factors of a sample sphere of steel; the sample had a diameter of 107 mm and measurements were taken at a tempreature of 4 K. Lower frequency Q's are fairly well fit by a Maxwell model with a viscoelastic ratio $h' = 0.3$, while higher frequency Q's require a Kelvin–Voigt with viscoelastic ratio $h' = 2.5$. In this case there more experimental points than theoretical due to mode splitting caused by the sphere suspension —see text.

often group around clearly distinct values, close to those predicted by theory. In such cases, therefore, the viscoelastic model is helpful to tell the different multiplet members from one another, an otherwise very difficult task.

Rather than characterising a given viscoelastic model by the two parameters λ' and μ', or α and β —see sections 3 and 4—, it is expedient for the purposes of this section to use a different parameter pair. This will be chosen as the Q of the lowest frequency toroidal mode, $Q_0 \equiv Q^T_{12}$, and the viscoelastic ratio h' of sections 3.2 and 4.2. The following expressions thus hold:

$$\text{Kelvin–Voigt:} \quad \begin{cases} Q^T_{nl} = \left(\dfrac{\omega^T_{12}}{\omega^T_{nl}}\right) Q_0 \\ Q^P_{nl} = \left(\dfrac{\omega^T_{12}}{\omega^P_{nl}}\right) f(k^P_{nl}R, h, h')\, Q_0 \end{cases} \tag{80a}$$

$$\text{Maxwell:} \quad \begin{cases} Q^T_{nl} = \left(\dfrac{\omega^T_{nl}}{\omega^T_{12}}\right) Q_0 \\ Q^P_{nl} = \left(\dfrac{\omega^P_{nl}}{\omega^T_{12}}\right) f(k^P_{nl}R, h, h')\, Q_0 \end{cases} \tag{80b}$$

obviously a rewrite of the corresponding ones in the respective sections.

In figure 2 we plot the measured versus theoretically calculated values of Q for

two sample spheres of Al 5056 and a 7 % alloy of CuAl —see more technical data in the caption to the figure. Agreement is quite good, when a Maxwell model is used, except for the higher frequency mode, reported in the case of CuAl. It appears that a better fit can be accomplished when a combination of both Maxwell and Kelvin–Voigt models is used, such as we see in figure 3, where Q's of a sample sphere of steel are considered. The latter plot contains more experimental points than theoretical, and we see here a clear example of frequency splittings due to symmetry breaking caused by suspension, normally a diametral or semi-diametral bore drilled across the sphere [31].

7. Conclusions

In this paper we have addressed the problem of whether it is possible to systematically characterise the linewidths or, equivalently, the Q's of the oscillation eigenmodes of a given spherical GW detector. To this end we have considered various phenomenological models, selected from the specialised literature on the subject, and solved the equations of motion in the cases of our interest. Different models are seen to predict different frequency dependences of the quality factors for the lower modes, which are the ones we have paid attention to, and the ones relevant for GW detection purposes. For example, in a Kelvin–Voigt solid the Q of a given mode appears to be inversely proportional to its frequency, while in a Maxwell solid it is directly proportional to it.

As we have seen, however, the behaviour of a given material is generally not well described by a *single* such model but rather by a suitable combination of e.g. Maxwell and Kelvin–Voigt models, depending on the range of frequencies considered. This is not too surprising, as these models are *phenomenological*, i.e., they are not based on a detailed analysis of the *physics* of the various dissipation mechanisms which are responsible for the loss processes [33], but on more or less plausible (though not unique) generalisations of the "friction is proportional to velocity" principle, which so accurately holds in a simple unidimensional harmonic oscillator.

In spite of these limitations, and also in spite of the experimental difficulties inherent in the determination of Q, it appears that our treatment of the problem gives results which can be considered quite good, in view of the above limitations. A significant bonus of the present analysis is the possibility it offers to identify nearby frequencies in the sphere spectrum by their distinct value of Q, as predicted by theory. This is useful in GW detection science since, for example, the *second* spheroidal quadrupole frequency, ω_{22}^{P}, is very important due to large cross section for GW energy absorption in that mode [1] yet it is very close (in frequency) to other modes with different Q's.

Improvements on the results presented in this paper are certainly possible. But we feel they would have to be based on a different methodology, with more emphasis *ab initio* on Solid State lore, and/or on Materials Science.

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Appendix A.

A well-behaved three-dimensional vector field $\mathbf{s}(\mathbf{x})$ can be expressed as the sum of an irrotational, $\mathbf{s}_l(\mathbf{x})$, and a divergence free, $\mathbf{s}_t(\mathbf{x})$, vector fields, respectively called the *longitudinal* and *transverse* components of $\mathbf{s}(\mathbf{x})$ [26]:

$$\mathbf{s}(\mathbf{x}) = \mathbf{s}_l(\mathbf{x}) + \mathbf{s}_t(\mathbf{x}) \ , \qquad \nabla{\cdot}\mathbf{s}_t = \nabla \times \mathbf{s}_l = 0 \tag{A.1}$$

We now replace this decomposition into equation (9) to find

$$\begin{aligned} \rho\ddot{T}(t)(\mathbf{s}_t + \mathbf{s}_l) = & \left[\mu T(t) + \mu'\dot{T}(t)\right] \nabla^2(\mathbf{s}_t + \mathbf{s}_l) \\ & + \left[(\lambda+\mu)\,T(t) + (\lambda'+\mu')\,\dot{T}(t)\right] \nabla\,(\nabla{\cdot}\mathbf{s}_l) \end{aligned} \tag{A.2}$$

Taking the rotational of this equation,

$$\nabla \times \left[\rho\ddot{T}\,\mathbf{s}_t - (\mu\,T + \mu'\,\dot{T})\,\nabla^2\mathbf{s}_t\right] = 0 \tag{A.3}$$

The vector between square brackets is thus divergence–free and irrotational, so it vanishes. We have therefore:

$$\nabla^2\mathbf{s}_t = \left\{ \frac{\rho\,\ddot{T}}{\mu\,T + \mu'\,\dot{T}} \right\} \mathbf{s}_t \tag{A.4}$$

Since the left hand side of the above equation does not depend on time, the term between braces in the right hand side must equal a (complex) constant, say $-\mathcal{K}^2$. Thus,

$$\nabla^2\mathbf{s}_t + \mathcal{K}^2\,\mathbf{s}_t = 0 \tag{A.5}$$

$$\mu\,T + \mu'\,\dot{T} + \mathcal{K}^{-2}\ddot{T} = 0 \tag{A.6}$$

An analogous procedure, after taking the divergence of equation (A.2), gives us the corresponding formulæ for the longitudinal part:

$$\nabla^2\mathbf{s}_l + \mathcal{Q}^2\,\mathbf{s}_l = 0 \tag{A.7}$$

$$(\lambda+2\mu)\,T + (\lambda'+2\mu')\,\dot{T} + \mathcal{Q}^{-2}\ddot{T} = 0 \tag{A.8}$$

where $\mathcal{Q}^2$ stands for another complex separation constant.

Appendix B.

We describe in this appendix the algebraic operations which lead to the solution to the eigenvalue problem in a viscoelastic sphere. Equations (2) ought to be solved, subject to the boundary conditions (5). The latter can be cast in explicit vector form:

$$(\lambda+\gamma\lambda')\,[\nabla{\cdot}\mathbf{s}(\mathbf{x})]\,\mathbf{n} + 2(\mu+\gamma\mu')\,(\mathbf{n}{\cdot}\nabla)\mathbf{s}(\mathbf{x}) + 2(\mu+\gamma\mu')\,\mathbf{n}\times[\nabla\times\mathbf{s}(\mathbf{x})] = 0 \tag{B.1}$$

The irrotational and divergence free components, $\mathbf{s}_t(\mathbf{x})$ and $\mathbf{s}_l(\mathbf{x})$, can be expressed by means of auxiliary functions $\phi(\mathbf{x})$ and $\psi(\mathbf{x})$:

$$\mathbf{s}_l(\mathbf{x}) = \mathcal{Q}^{-1} C_0 \, \nabla \, \phi(\mathbf{x}) \tag{B.2}$$

$$\mathbf{s}_t(\mathbf{x}) = i\mathcal{K}^{-1} C_1 \, \nabla \times \mathbf{L}\psi(\mathbf{x}) + i C_2 \, \mathbf{L}\psi(\mathbf{x}) \tag{B.3}$$

where C_0, C_1 and C_2 are (so far) undetermined integration constants, and $\mathbf{L} \equiv -i\mathbf{x}\times\nabla$ is the "angular momentum" operator. Upon substitution of (B.2)-(B.3) into (2) it is readily seen that the functions ϕ and ψ are themselves also solutions to corresponding Helmholtz equations,

$$\nabla^2\phi(\mathbf{x}) + \mathcal{Q}^2\phi(\mathbf{x}) = 0 \ , \quad \text{and} \quad \nabla^2\psi(\mathbf{x}) + \mathcal{K}^2\psi(\mathbf{x}) = 0 \tag{B.4}$$

They have therefore the general form, using spherical coordinates (r,θ,φ) for the vector $\mathbf{x}$,

$$\phi(\mathbf{x}) = j_l(\mathcal{Q}r)\, Y_{lm}(\theta,\varphi) \ , \quad \text{and} \quad \psi(\mathbf{x}) = j_l(\mathcal{K}r)\, Y_{lm}(\theta,\varphi) \tag{B.5}$$

where j_l are *spherical* Bessel functions of the first kind and Y_{lm} are spherical harmonics. The solutions (B.5) are those possessing regularity properties in the whole interior and boundary of the solid. We thus have

$$\nabla\phi \quad = \frac{d\, j_l(\mathcal{Q}r)}{dr}\, Y_{lm}(\theta,\varphi)\, \mathbf{n} - \frac{j_l(\mathcal{Q}r)}{r}\, i\mathbf{n}\times\mathbf{L}Y_{lm}(\theta,\varphi) \tag{B.6}$$

$$\begin{aligned} \nabla\times\mathbf{L}\psi \ &= \ -\, l(l+1)\, \frac{j_l(\mathcal{K}r)}{r}\, Y_{lm}(\theta,\varphi)\, \mathbf{n} \\ &+ \ \left[\frac{j_l(\mathcal{K}r)}{r} + \frac{d}{dr} j_l(\mathcal{K}r)\right]\, i\mathbf{n}\times\mathbf{L}Y_{lm}(\theta,\varphi) \end{aligned} \tag{B.7}$$

$$\mathbf{L}\psi \quad = j_l(\mathcal{K}r)\, i\mathbf{L}Y_{lm}(\theta,\varphi) \tag{B.8}$$

These expressions ought to be substituted now into (B.2)-(B.3), and then into (B.1) —recall that $\mathbf{s} = \mathbf{s}_t + \mathbf{s}_l$. It is found thet these are equivalent to the following *homogeneous linear system*

$$\begin{pmatrix} \beta_4\left(\mathcal{Q}R, \frac{\lambda+\gamma\lambda'}{\mu+\gamma\mu'}\right) & -l(l+1)\frac{\mathcal{K}}{\mathcal{Q}}\beta_1(\mathcal{K}R) & 0 \\ -\beta_1(\mathcal{Q}R) & \frac{\mathcal{K}}{\mathcal{Q}}\,\beta_3(\mathcal{K}R) & 0 \\ 0 & 0 & -\frac{\mathcal{K}}{\mathcal{Q}}\,\mathcal{K}R\,\beta_1(\mathcal{K}R) \end{pmatrix} \begin{pmatrix} C_0 \\ C_1 \\ C_2 \end{pmatrix} = 0 \tag{B.9}$$

where

$$\beta_0(z) \quad \equiv \frac{j_l(z)}{z^2} \tag{B.10}$$

$$\beta_1(z) \quad \equiv \frac{d}{dz}\left[\frac{j_l(z)}{z}\right] \tag{B.11}$$

$$\beta_2(z) \quad \equiv \frac{d^2}{dz^2}\left[j_l(z)\right] \tag{B.12}$$

$$\beta_3(z) \quad \equiv \tfrac{1}{2}\,\beta_2(z) + \left\{\tfrac{l(l+1)}{2} - 1\right\}\,\beta_0(z) \tag{B.13}$$

$$\beta_4(z,A) \equiv \beta_2(z) - \frac{A}{2} j_l(z) \tag{B.14}$$

The system (B.9) is to be satisfied by the constants C_0, C_1 and C_2, but has no meaningful solution unless the system matrix is *singular*, i.e., if its determinant vanishes. Therefore

$$\beta_1(\mathcal{K}R)\ \det\begin{pmatrix} \beta_4\left(\mathcal{Q}R, \frac{\lambda+\gamma\lambda'}{\mu+\gamma\mu'}\right) & -l(l+1)\frac{\mathcal{K}}{\mathcal{Q}}\beta_1(\mathcal{K}R) \\ -\beta_1(\mathcal{Q}R) & \frac{\mathcal{K}}{\mathcal{Q}}\beta_3(\mathcal{K}R) \end{pmatrix} = 0 \qquad \text{(B.15)}$$

This is an equation for the parameter γ, on which $\mathcal{K}$ and $\mathcal{Q}$ depend through equations (14*a*) and (14*b*). Clearly, there are *two* families of solutions, or *eigenmodes*, to (B.15) associated to the vanishing of either of the two factors in its lhs, i.e., $\beta_1(\mathcal{K}R)$ *or* the determinant of the displayed 2×2 matrix. They are called *toroidal* and *spheroidal* solutions, respectively.

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